# ON THE UNIT DOT PRODUCT GRAPH OF A COMMUTATIVE RING

Correction:
Theorem 6.1 (1, 2, 3) and Theorem 6.2 (1, 2, 3): The set D is a dominating set and it is minimum in the sense that if F is another dominating set and F is a subset of D, then D = F. However, we cannot claim that D is a minimum dominating set. So the given dominating number \gamma need not be true.

Theorem 6.1 (4) and Theorem 6.2 (4): The results are OK .

by

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## Approval Signatures

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# Abstract

In 2015, Ayman Badawi (Badawi, 2015) introduced the *dot product graph* associated to a commutative ring $A$. Let $A$ be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). We recall from (Badawi, 2015) that *total dot product graph* of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, ..., 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$). Let $Z(R)$ denotes the set of all zero-divisors of $R$. Then the *zero-divisor dot product graph* of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, ..., 0)\}$. Let $U(R)$ denotes the set of all units of $R$. Then the *unit dot product graph* of $R$ is the induced subgraph $UD(R)$ of $TD(R)$ with vertices $U(R)$. Let $n \geq 2$ and $A = Z_n$. The main goal of this thesis is to study the structure of $UD(R = A \times A)$.

**Search Terms**: Total dot product graphs, zero dot product graphs, dominating sets, domination number.

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## 1. Introduction

Let $R$ be a commutative ring with $1 \neq 0$. Then $Z(R)$ denotes the set of zero-divisors of $R$ and the group of units of $R$ will be denoted by $U(R)$. As usual $Z_n$, denotes the ring of integers modulo $n$. The nonzero elements of $S \subseteq R$ will be denoted by $S^*$. Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles (D. Anderson, Axtell, & Stickles, 2010) and (H. Maimani, Pouranki, Tehranian, & Yassemi, 2011). For example, as in (D.F. & Livingston, 1999), the *zero-divisor graph* of $R$ is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. This concept is due to Beck (Beck, 1988), who let all the elements of $R$ be vertices and was mainly interested in colorings. The zero-divisor graph of a ring $R$ has been studied extensively by many authors, for example see((Akbari, Maimani, & Yassemi, 2003)-(D. D. Anderson & Naseer, 1993), (D. Anderson & Badawi, 2008a), (Axtel, Coykendall, & Stickles, 2005)-(Axtel & Stickles, 2006), (Chiang-Hsieh, Smith, & Wang, 2010)-(DeMeyer, Greve, Sabbaghi, & Wang, 2010), (H. R. Maimani, Pournaki, & Yassemi, 2006)-(Smith, 2007), (Wickham, 2008)). We recall from (D. Anderson & Badawi, 2008b), the *total graph* of $R$, denoted by $T(\Gamma(R))$ is the (undirected) graph with all elements of $R$ as vertices, and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$. The total graph (as in (D. Anderson & Badawi, 2008b)) has been investigated in (Akbari, Kiani, Mohammadi, & Moradi, 2009), (Akbari, Jamaali, & Seyed Fakhari, 2009), (Akbari, Aryapoor, & Jamaali, 2012), (?, ?), (H. Maimani et al., 2011), (H. Maimani, Wickham, & Yassemi, 2012), (Pucanović & Petrović, 2011), (Chelvam & Asir, 2013c) and (Shekarriz, Shiradareh Haghighi,

& Sharif, 2012); and several variants of the total graph have been studied in (Abbasi & Habib, July 2001), (D. Anderson & Badawi, 2012), (D. Anderson & Badawi, 2013), (D. Anderson, Fasteen, & LaGrange, 2012), (Atani & Habibi, 2011), (Barati, Khashyarmanesh, Mohammadi, & Nafar, 2012), (Chelvam & Asir, 2013b), (Chelvam & Asir, 2011), (Chelvam & Asir, 2012), (?, ?), (Chelvam & Asir, 2013a), and (Khashyarmanesh & Khorsandi, 2012). Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. In 2014, Badawi (Badawi, 2014) introduced the annihilator graph of $R$. We recall from (Badawi, 2014) that the annihilator graph of $R$ is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of the classical zero-divisor of $R$ is an edge (path) of $AG(R)$. For Further investigations of $AG(R)$, see (Afkhami, Khashyarmanesh, & Sakhdari, 2015), and (Visweswaran & Patel, 2014).

In 2015, Badawi (Badawi, 2015) introduced the *dot product graph* associated to a commutative ring $A$. Let $A$ be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). We recall from [1] that *total dot product graph* of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, ..., 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$). Let $Z(R)$ denotes the set of all zero-divisors of $R$. Then the *zero-divisor dot product graph* of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, ..., 0)\}$. Let $U(R)$ denotes the set of all units of $R$. Then the *unit dot product graph* of $R$ is the induced subgraph $UD(R)$ of $TD(R)$ with vertices $U(R)$. Let $n \geq 2$ and $A = Z_n$. The main goal of this thesis is to study the structure of $UD(R = A \times A)$. Let $G$ be a graph with $V$ as its set of vertices. We recall that a subset $S \subseteq V$ is called a *dominating set of $G$* if every vertex in $V$ is either in $S$ or is adjacent to a vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality among the dominating sets of $G$. If $A = Z_n$

and $R = Z_n \times \cdots \times Z_n$ ($m$ times, where $m < \infty$), then the domination numbers of $TD(R)$ and $ZD(R)$ are determined. Furthermore, the domination number of $UD(Z_n \times Z_n)$ is determined.

Let $G$ be a graph. Two vertices $v_1, v_2$ of $G$ are said to be *adjacent* in $G$ if $v_1, v_2$ are connected by an edge (line segment) of $G$ and we write $v_1 - v_2$. A finite sequence of edges from a vertex $v_1$ of $G$ to a vertex $v_2$ of $G$ is called a *path* of $G$ and we write $v_1 - a_1 - a_2 - \cdots - a_k - v_2$, where $k < \infty$ and the $a_i$, $1 \leq i \leq k$, are some distinct vertices of $G$. Hence it is clear that every edge of $G$ is a path of $G$, but not every path of $G$ is an edge of $G$. We say that $G$ is *connected* if there is a path between any two distinct vertices of $G$. At the other extreme, we say that $G$ is *totally disconnected* if no two vertices of $G$ are adjacent. We denote the complete graph on $n$ vertices by $K_n$ (recall that a graph $G$ is called complete if every two vertices of $G$ are adjacent) and the complete bipartite graph on $m$ and $n$ vertices by $K_{m,n}$ (we allow $m$ and $n$ to be infinite cardinals, recall that $K_{m,n}$ is the graph with two sets of vertices, say $V_1, V_2$ such that $|V_1| = n, |V_2| = m$, $V_1 \cap V_2 = \emptyset$, every two vertices in $V_1$ are not adjacent, every two vertices in $V_2$ are not adjacent, and every vertex in $V_1$ is adjacent to every vertex in $V_2$). We will sometimes call a $K_{1,n}$ a *star graph*. We say that two (induced) subgraphs $G_1$ and $G_2$ of $G$ are *disjoint* if $G_1$ and $G_2$ have no common vertices and no vertex of $G_1$ (resp., $G_2$) is adjacent (in $G$) to any vertex not in $G_1$ (resp., $G_2$). A general reference for graph theory is (Bollaboás, 1979).

## 2. The Structure of $UD(R = A \times A)$ When $A$ Is a Field

Let $p$ be a positive prime number, $n \geq 1$. Then $A = GF(p^n)$ denotes a finite field with $p^n$ elements. Let $R = A \times A$. Then $TD(R)$ is not connected by (Badawi, 2015, Theorem 2.1). The first two results give a complete description of the structure of $UD(R)$ and $TD(R)$.

**Theorem 2.1.** *Let $n \geq 1$, $m = 2^n - 1$ and $R = GF(2^n) \times GF(2^n)$. Then*

1. *$ZD(R) = \Gamma(R) = K_{m,m}$.*
2. *$UD(R)$ is the union of one $K_m$ and $(2^{(n-1)} - 1)$ disjoint $K_{m,m}$'s.*
3. *$TD(R)$ is the union of one $K_m$ and $2^{(n-1)}$ disjoint $K_{m,m}$'s.*

*Proof.* **(1)**. The result is clear by (Badawi, 2015, Theorem 2.1) and (D. Anderson & Mulay, 2007, Theorem 2.2).

**(2)**. Let $A = GF(2^n)$. Then $R = A \times A$. Let $v_1, v_2 \in U(R)$. Since $R$ is a vector space over $A$, $v_1 = u(1, a) \in R$ and $v_2 = v(1, b) \in R$ for some $u, v, a, b \in A^*$. Hence $v_1$ is adjacent to $v_2$ if and only if $v_1.v_2 = uv + uvab = 0$ in $A$ if and only if $b = -a^{-1} = a^{-1}$ in $A$ . Thus for each $a \in U(A) = A^*$, let $X_a = \{u(1, a) \mid u \in A^*\}$ and $Y_a = \{u(1, a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Y_a| = 2^n - 1$. Let $a = 1$. Since $char(A) = char(R) = 2$, $X_a = Y_a$ and the dot product of every two distinct vertices in $X_a$ is zero. Thus every two distinct vertices in $X_a$ are adjacent. Thus the vertices in $X_a$ form the graph $K_m$ that is a complete subgraph of $TD(R)$. Let $a \in U(A)$ such that $a \neq 1$. Since $a^2 \neq 1$ for each $a \in U(A) \setminus \{1\}$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. Since $char(A) = char(R) = 2$, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $(2^n - 2)/2 = 2^{n-1} - 1$ disjoint complete bi-partite

$K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the union of one complete subgraph $K_m$ and $(2^{n-1} - 1)$ disjoint complete bi-partite $K_{m,m}$ subgraphs.

**(3)**. The claim follows from (1) and (2).

□

**Theorem 2.2.** *Let $p \geq 3$ be a positive prime integer, $n \geq 1$, $m = p^n - 1$, and let $R = GF(p^n) \times GF(p^n)$. Then*

1. *$ZD(R) = \Gamma(R) = K_{m,m}$.*
2. *If $4 \nmid m$, then $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.*
3. *If $4 \mid m$, then $UD(R)$ is the union of two $K_m$'s and $(m-2)/2$ disjoint $K_{m,m}$'s.*
4. *If $4 \nmid m$, then $TD(R)$ is the union of $(m+2)/2$ disjoint $K_{m,m}$'s.*
5. *If $4 \mid m$, then $TD(R)$ is the union of two $K_m$'s and $m/2$ disjoint $K_{m,m}$'s.*

*Proof.* **(1)**. The result is clear by (Badawi, 2015, Theorem 2.1) and (D. Anderson & Mulay, 2007, Theorem 2.2).

**(2)** Let $A = GF(p^n)$. Then $R = A \times A$. Let $v_1, v_2 \in U(R)$. Since $R$ is a vector space over $A$, $v_1 = u(1, a) \in R$ and $v_2 = v(1, b) \in R$ for some $u, v, a, b \in A^*$. Hence $v_1$ is adjacent to $v_2$ if and only if $v_1.v_2 = uv + uvab = 0$ in $A$ if and only if $b = -a^{-1}$ in $A$. Thus for each $a \in U(A) = A^*$, let $X_a = \{u(1, a) \mid u \in A^*\}$ and $Y_a = \{u(1, a^{-1}) \mid u \in A^*\}$. Since $R$ is a vector space over $A$, for each $a \in U(A) = A^*$, let $X_a = \{u(1, a) \mid u \in A^*\}$ and $Y_a = \{u(1, -a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Y_a| = m = p^n - 1$. Since $4 \nmid m$, $U(A) = A^*$ has no elements of order 4. Thus $a^2 \neq -1$ for each $a \in U(A)$. Hence $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction of $X_a$ and $Y_a$, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph

$K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $m/2$ disjoint complete bi-partite $K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.

**(3)**. Note that $|U(A)| = m$. Since $U(A) = A^*$ is cyclic and $4 \mid m$, $U(A)$ has exactly one subgroup of order 4. Thus $U(A)$ has exactly two elements of order 4, say $b, c$. Since $a \in U(A)$ is of order 4 if and only if $a^2 = -1$, it is clear that $x^2 = -1$ for some $x \in U(A)$ if and only if $x = b, c$. Let $X_b = \{u(1, b) \mid u \in U(A)\}$ and let $X_c = \{u(1, c) \mid u \in U(A)\}$. It is clear that $|X_b| = |X_c| = m$. Let $H = \{b, c\}$. Then the dot product of every two distinct vertices in $X_h$ is zero for each $h \in H$. Thus every two distinct vertices in $X_h$ are adjacent for every $h \in H$. Thus for each $h \in H$, the vertices in $X_h$ form the graph $K_m$ that is a complete subgraph of $TD(R)$. Let $a \in U(A) \setminus H$, $X_a = \{u(1, a) \mid u \in A^*\}$, and $Y_a = \{u(1, -a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Ya| = m$. Since $a \notin H$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are $(m-2)/2$ disjoint $K_{m,m}$ subgraphs. Hence $UD(R)$ is the union of two $K_m$'s and $(m-2)/2$ disjoint $K_{m,m}$'s.

**(4)**. The claim follows from (1) and (2).

**(5)**. The claim follows from (1) and (3).

$\square$

In view of Theorem 2.2, we have the following corollary.

**Corollary 2.3.** Let $p \geq 3$ be a prime positive integer, and let $R = \mathbb{Z}_p \times \mathbb{Z}_p$. Then

1. $ZD(R) = \Gamma(R) = K_{p-1,p-1}$ .

2. If $4 \nmid p-1$, then $UD(R)$ is the union of $(p-1)/2$ disjoint $K_{p-1,p-1}$.

3. If $4 \mid p-1$, then $UD(R)$ is the union of two disjoint $K_{p-1}$'s and $(p-3)/2$ disjoint $K_{p-1,p-1}$'s.

4. If $4 \nmid p-1$, then $TD(R)$ is the union of $(p+1)/2$ disjoint $K_{p-1,p-1}$'s.

5. If $4 \mid p-1$, then $TD(R)$ is the union of two disjoint $K_{p-1}$'s and $(p-1)/2$ disjoint $K_{p-1,p-1}$'s.

**Example 2.4.** Let $A = \dfrac{Z_2[X]}{(X^2+X+1)}$. Then $A$ is a finite field with 4 elements. Let $v = X + (X^2+X+1) \in A$. Since $(A^*, .)$ is a cyclic group and $A^* =< v >$, we have $A = \{0, v, v^2, v^3 = 1 + (X^2+X+1)\}$ . Let $R = A \times A$. Then the $UD(R)$ is the union of one $K_3$ and one $K_{3,3}$ by Theorem 2.1(1). The following is the graph of $UD(R)$.

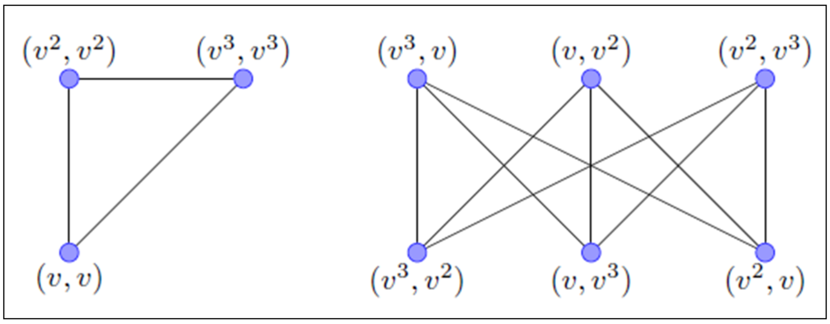


Fig. 2.1: The unit dot product graph of the ring $A \times A$, where $A$ is a field with 4 elements

**Example 2.5.** Let $A = Z_5$ and $R = A \times A$. Then the $UD(R)$ is the union of two disjoint $K_4$ and one $K_{4,4}$ by Corollary 2.3(3). The following is the graph of $UD(R)$.

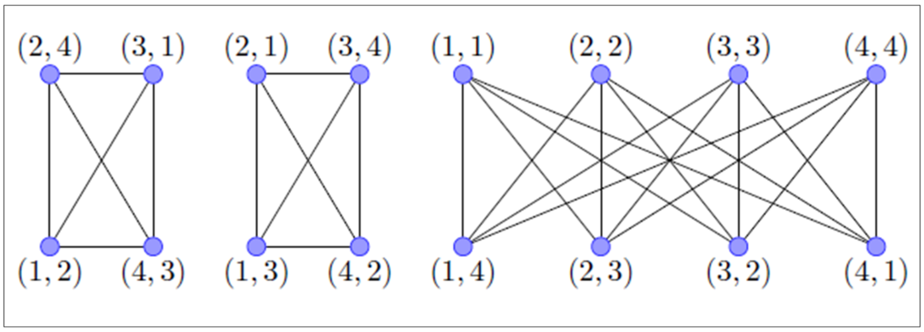


Fig. 2.2: The unit dot product graph of the ring $Z_5 \times Z_5$

## 3. Unit Dot Product Graph of $R = Z_n \times Z_n$

Let $n > 1$ and write $n = p_1^{k_1} \cdots p_m^{k_m}$, where the $p_i$'s are distinct prime positive integers. Then $U(Z_n) = \{1 \leq a < n$ |a is an integer and $gcd(a, n) = 1\}$. It is known that $U(Z_n)$ is a group under multiplication module $n$ and $|U(Z_n)| = \phi(n) = (p_1 - 1)p_1^{k_1-1}(p_2 - 1)p_2^{k_2-1} \cdots (p_m - 1)p_m^{k_m-1}$.

If $n \geq 3$, then it is clear that $\phi(n)$ is an even integer. In the next result, we give an alternative proof of this fact.

**Proposition 3.1.** Let $n$ be an integer such that $n \geq 3$. Then $\phi(n)$ is an even integer.

*Proof.* Let $k \in U(Z_n)$. It is clear that $gcd(n - k, n) = 1$ and thus $n - k \in U(Z_n)$. It is also clear that $k, n - k$ are distinct elements in $U(Z_n)$. Thus all numbers in $U(Z_n)$ can be put into pairs. Hence if $n \geq 3$, then $\phi(n)$ is an even integer. $\square$

The following lemma is needed.

**Lemma 3.2.** Let $n$ be a positive integer and write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the $p_i$'s are distinct prime positive integers. Then

1. If $4 \mid n$, then $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$.
2. If $4 \nmid n$, then $x^2 \equiv n - 1 \pmod{n}$ has a solution in $U(Z_n)$ if and only if $4 \mid (p_i - 1)$ for each odd prime factor $p_i$ of $n$. Furthermore, if $x^2 \equiv n-1$ $(mod\ n)$ has a solution in $U(Z_n)$, then it has exactly $2^{r-1}$ distinct solutions in $U(Z_n)$ if $n$ is even and it has exactly $2^r$ distinct solutions in $U(Z_n)$ if $n$ is odd.

*Proof.* **(1)**. Suppose that $4 \mid n$. Then $n \geq 4$. Since $4 \nmid (n - 2)$, $n - 1 \not\equiv 1$ $(mod\ 4)$ and thus $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$ by (LeVeque, 1977, Theorem 5.1).

**(2)**. Suppose that $4 \nmid n$. Then $a^2 \equiv n-1\ (mod\ n)$ for some $a \in U(Z_n)$ if and only if $a^2 \equiv n-1\ (mod\ p_i)$ for each odd prime factor $p_i$ of $n$ by (LeVeque, 1977, Theorem 5.1). Thus $a^2 \equiv n-1\ (mod\ n)$ for some $a \in U(Z_n)$ if and only if $(a\ mod\ p_i)^2 \equiv p_i - 1\ (mod\ p_i)$ for each odd prime factor $p_i$ of $n$. Since $U(Z_{p_i}) = Z^*_{p_i} = \{1, ..., p_i - 1\}$ for each prime factor $p_i$ of $n$, we have $|U(Z_{p_i})| = p_i - 1$. For each $x \in U(Z_{p_i})$, $1 \leq i \leq r$, let $|x|$ denotes the order of $x$ in $U(Z_{p_i})$. Let $p_i$, $1 \leq i \leq r$, be an odd prime factor of $n$. Since $|p_i - 1| = 2$ in $U(Z_{p_i})$, $b^2 = p_i - 1$ in $U(Z_{p_i})$ for some $b \in U(Z_{p_i})$ if and only if $|b| = 4$ in $U(Z_{p_i})$. Since $|U(Z_{p_i})| = p_i - 1$, we conclude that $b^2 = p_i - 1$ in $U(Z_{P_i})$ for some $b \in U(Z_{p_i})$ if and only if $4 \mid (p_i - 1)$. Thus $x^2 \equiv n-1 \pmod{n}$ has a solution in $U(Z_n)$ if and only if $4 \mid (p_i - 1)$ for each odd prime $p_i$ factor of $n$. Suppose that $x^2 \equiv n-1\ (mod\ n)$ has a solution in $U(Z_n)$. We consider two cases:

**Case 1**. Suppose that $n$ is an even integer. Then there are exactly $r-1$ distinct odd prime factors of $n$. Since $4 \nmid n$, $x^2 \equiv n-1\ (mod\ n)$ has exactly $2^{r-1}$ distinct solutions in $U(Z_n)$ by (LeVeque, 1977, Theorem 5.2).

**Case 2**. Suppose that $n$ is an odd integer. Then there are exactly $r$ distinct odd prime factors of $n$. Thus $x^2 \equiv n-1\ (mod\ n)$ has exactly $2^r$ distinct solutions in $U(Z_n)$ by (LeVeque, 1977, Theorem 5.2). □

Let $A = Z_n$, where $n$ is not prime. Then $TD(A \times A)$ is connected by (Badawi, 2015, Theorem 2.3 ). In the following result, we show that $UD(A \times A)$ is disconnected, and we give a complete description of the structure of $UD(A \times A)$.

**Theorem 3.3.** *Let $n \geq 2$ be an integer, $R = Z_n \times Z_n$ and $\phi(n) = m$. Then*

1. *If $4 \mid n$, then $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.*

2. *If $4 \nmid n$ and $4 \nmid (p_i - 1)$ for at least one of the $p_i$'s in the prime factorization of $n$, then $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.*

*3. If $4 \nmid n$ and $4 \mid (p_i - 1)$ for all the odd $p_i$'s in the prime factorization of $n$, then we consider the two cases:*

**Case I**. *If $n$ is even, then $UD(R)$ is a union of $(m/2) - 2^{r-2}$ disjoint $K_{m,m}$'s and $2^{r-1}$ disjoint $K_m$'s.*

**Case II**. *If $n$ is odd, then $UD(R)$ is a union of $(m/2) - 2^{r-1}$ disjoint $K_{m,m}$'s and $2^r$ disjoint $K_m$'s.*

*Proof.* Let $A = Z_n$. Then $R = A \times A$. Note that $UD(R)$ has exactly $m^2$ vertices. Let $v_1, v_2 \in U(R)$. Since $R$ is a vector space over $A$, $v_1 = u(1, a) \in R$ and $v_2 = v(1, b) \in R$ for some $u, v, a, b \in U(A)$. Hence $v_1$ is adjacent to $v_2$ if and only if $v_1.v_2 = uv + uvab = 0$ in $A$ if and only if $b = -a^{-1}$ in $A$. Thus for each $a \in U(A)$, let $X_a = \{u(1, a) \mid u \in U(A)\}$ and $Y_a = \{u(1, -a^{-1}) \mid u \in U(A)\}$. It is clear that $|X_a| = |Y_a| = m$.

**(1)**. Since $4 \mid n$, $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$ by Lemma 3.2(1). Hence $X_a \cap Y_a = \emptyset$. It is clear that every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction of $X_a$ and $Y_a$, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $m/2$ disjoint complete bi-partite $K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.

**(2)**. Write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the $p_i$'s are distinct prime positive integers. Since $4 \nmid n$ and $4 \nmid (p_i - 1)$ for at least one of the $p_i$'s, $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$ by Lemma 3.2. Thus by the same argument as in (1), $UD(R)$ is the union of $m/2$ disjoint $K_{m,m}$'s.

**(3)**. Write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the $p_i$'s are distinct prime positive integers. Suppose that $4 \nmid n$ and $4 \mid p_i - 1$ for all the odd $p_i$'s in the prime factorization of $n$. Let $B = \{b \in U(Z_n) \mid b^2 = n - 1 \text{ in } U(Z_n)\}$ and $C = \{c \in$

$U(Z_n) \mid c^2 \neq n-1$ in $U(Z_n)\}$. We consider two cases:

**Case I**. Suppose that $n$ is even. Then $|B| = 2^{r-1}$ by Lemma 3.2(2) and hence $|C| = m - 2^{r-1}$. For each $a \in B$, we have $X_a = Y_a$ and hence the dot product of every two distinct vertices in $X_a$ is zero. Thus the vertices in $X_a$ form the graph $K_m$ that is a complete subgraph of $TD(R)$. Hence $UD(Z_n)$ has exactly $2^{r-1}$ disjoint $K_m$'. For each $a \in C$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. Thus $UD(Z_n)$ has exactly $\frac{m-2^{r-1}}{2} = \frac{m}{2} - 2^{r-2}$ disjoint $K_{m,m}$'s.

**Case II**. Suppose that $n$ is odd. Then $|B| = 2^r$ by Lemma 3.2(2) and hence $|C| = m - 2^r$. For each $a \in B$, we have $X_a = Y_a$ and hence the dot product of every two distinct vertices in $X_a$ is zero. Thus the vertices in $X_a$ form the graph $K_m$ that is a complete subgraph of $TD(R)$. Hence $UD(Z_n)$ has exactly $2^r$ disjoint $K_m$'. For each $a \in C$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. Thus $UD(Z_n)$ has exactly $\frac{m-2^r}{2} = \frac{m}{2} - 2^{r-1}$ disjoint $K_{m,m}$'s. $\square$

Recall that a graph $G$ is called *completely disconnected* if every two vertices of $G$ are not connected by an edge in $G$.

**Theorem 3.4.** *Let $n \geq 4$ be an even integer, and let $R = Z_n \times Z_n \times ....Z_n$ (k times), where $k$ is an odd positive integer. Then $UD(R)$ is completely disconnected.*

*Proof.* Let $x = (x_1, ..., x_k), y = (y_1, ..., y_k) \in U(R)$. Then $x_i, y_i \in U(Z_n)$ for every i, $1 \leq i \leq k$. Since $n$ is an even integer, $x_i$ and $y_i$ are odd integers for every $i$, $1 \leq i \leq k$. Hence, since $k$ is an odd integer, $x_1y_1 + \cdots + x_ky_k$ is an odd integer, and thus $x_1y_1 + \cdots + x_ky_k \neq 0$ in $Z_n$, since $n$ is even. Thus $UD(R)$ is completely disconnected. $\square$

**Theorem 3.5.** *Let $n \geq 4$ be an even integer, and let $R = Z_n \times Z_n$. Then the vertex $(n/2, n/2)$ in $ZD(R)$ is adjacent to every vertex in $UD(R)$.*

*Proof.* It is clear that $(\frac{n}{2}, \frac{n}{2})$ is a vertex of $ZD(R)$. Let $u \in U(Z_n)$. Since $n$ is even, $u$ is an odd integer. Thus $u - 1 = 2m$ for some integer $m$. Hence $\frac{n}{2}(u-1) = \frac{n}{2}(2m) = mn = 0 \in Z_n$. Thus $\frac{n}{2}u = \frac{n}{2}$ in $Z_n$. Now let $(a, b) \in U(R)$. Then $a, b \in U(Z_n)$ are odd integers. Hence $(a, b)(\frac{n}{2}, \frac{n}{2}) = \frac{n}{2} + \frac{n}{2} = n = 0 \in Z_n$. Thus the vertex $(n/2, n/2)$ in $ZD(R)$ is adjacent to every vertex in $UD(R)$. $\square$

**Example 3.6.** Let $A = Z_8$ and $R = A \times A$. Then the $UD(R)$ is the union of two disjoint $K_{4,4}$ by Theorem 3.3(1). The following is the graph of $UD(R)$.

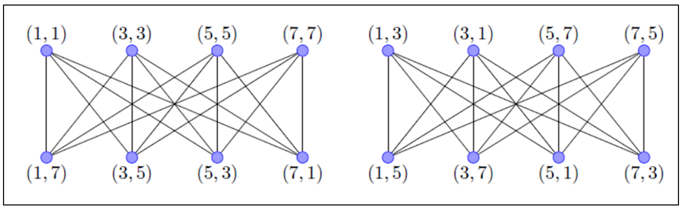


Fig. 3.1: The unit dot product graph of the ring $Z_8 \times Z_8$

**Example 3.7.** Let $A = Z_{10}$ and $R = A \times A$. Then the $UD(R)$ is the union of two disjoint $K_4$ and one $K_{4,4}$ by Theorem 3.3(3, case I). The following is the graph of $UD(R)$.

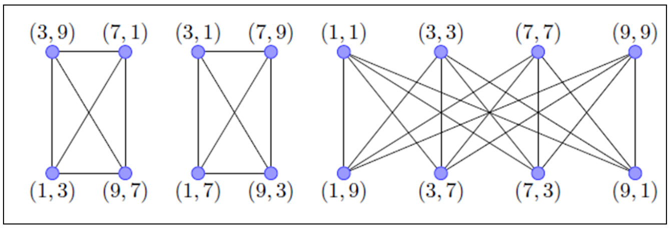


Fig. 3.2: The unit dot product graph of the ring $Z_{10} \times Z_{10}$

## 4. Subgraphs of the Zero-Divisor Dot Product Graph of $Z_n \times Z_n$

For an integer $n \geq 2$, let $R_1 = \{(u_1, z_1) \mid u_1 \in U(Z_n) \text{ and } z_1 \in Z(Z_n)\}$ and $R_2 = \{(z_2, u_2) \mid u_2 \in U(Z_n) \text{ and } z_2 \in Z(Z_n)\}$. It is clear that $R_1 \subset Z(Z_n \times Z_n)$ and $R_2 \subset Z(Z_n \times Z_n)$. In this section, we study the induced subgraph $ZD(R_1 \cup R_2)$ of $ZD(Z_n \times Z_n)$ with vertices $R_1 \cup R_2$.

**Theorem 4.1.** *Let $R = Z_n \times Z_n$ and $\phi(n) = m$. Then*

1. *If $n$ is prime, then $ZD(R_1 \cup R_2) = ZD(Z_n \times Z_n) = \Gamma(R) = K_{n-1,n-1}$.*
2. *If $n$ is not prime, then $ZD(R_1 \cup R_2)$ is the union of of $(n-m)$ disjoint $K_{m,m}$'s.*

*Proof.* **(1)**. Suppose that $n$ is prime. Then it is clear that $R_1 \cup R_2 = Z(Z_n \times Z_n)$. If $n = 2$, then it is trivial to see that $ZD(R_1 \cup R_2) = ZD(Z_n \times Z_n) = \Gamma(R) = K_{1,1}$. If $n \geq 3$, then the claim is clear by Corollary 2.3(1).

**(2)**. Let $A = Z_n$. Suppose that $n$ is not prime. It is clear that every two vertices in $R_i$ are not adjacent for every $i \in \{1, 2\}$. Let $v_1 \in R_1$ and $v_2 \in R_2$. Then $v_1 = u(1, a) \in R_1$ and $v_2 = v(b, 1) \in R_2$ for some $u, v \in U(A)$ and some $a, b \in Z(A)$. Then $v_1$ is adjacent to $v_2$ if and only if $v_1.v_2 = uvb + uva = 0$ in $A$ if and only if $b = -a$ in $A$. Hence for each $a \in Z(A)$, let $X_a = \{u(1, a) \mid u \in U(A)\}$ and $Y_a = \{u(-a, 1) \mid u \in U(A)\}$. It is clear that $|X_a| = |Y_a| = m$. For each $a \in Z(A)$, $X_a \cap Y_a = \emptyset$, every two distinct vertices in $X_a$ are not adjacent, and every two distinct vertices in $Y_a$ are not adjacent. By construction, it is clear that every vertex in $X_a$ is adjacent to every vertex in $Y_a$. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $ZD(R)$. Since $|R_1| = |R_2| = m(n-m)$ and $R_1 \cap R_2 = \emptyset$, we have $|R_1 \cup R_2| = 2m(n-m)$. Thus $ZD(R_1 \cup R_2)$ is the union of of $(n-m)$ disjoint $K_{m,m}$'s. $\square$

## 5. Equivalence Dot Product Graph

Let $A = Z_n$ and $R = A \times A$. Define a relation $\sim$ on $U(R)$ such that $x \sim y$, where $x, y \in U(R)$, if $x = (c, c)y$ for some $(c, c) \in U(R)$. It is clear that $\sim$ is an equivalence relation on $U(R)$. If $S$ is an equivalence class of $U(R)$, then there is an $a \in U(A)$ such that $S = (\overline{(1, a)} = \{u(1, a) \mid u \in U(Z_n)\}$. Let $E(U(R))$ be the set of all distinct equivalence classes of $U(R)$. We define the *equivalence unit dot product graph* of $U(R)$ to be the (undirected) graph $EUD(R)$ with vertices $E(U(R))$, and two distinct vertices $X$ and $Y$ are adjacent if and only if $a \cdot b = 0 \in A$ for every $a \in X$ and every $b \in Y$ (where $a \cdot b$ denote the normal dot product of $a$ and $b$). We have the following results.

**Theorem 5.1.** *Let $n \geq 1$, $m = 2^n - 1$ and $R = GF(2^n) \times GF(2^n)$. Then $EUD(R)$ is the union of one $K_1$ and $(2^{(n-1)} - 1)$ disjoint $K_{1,1}$'s.*

*Proof.* Let $A = GF(2^n)$. For each $a \in U(A)$, let $X_a$ and $Y_a$ be as in the proof of Theorem 2.1. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude that each $K_m$ of $UD(R)$ is a $K_1$ of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 2.1. $\square$

**Theorem 5.2.** *Let $p \geq 3$ be a positive prime integer, $n \geq 1$, $m = p^n - 1$, and let $R = GF(p^n) \times GF(p^n)$. Then*

1. *If $4 \nmid m$, then $EUD(R)$ is the union of $m/2$ disjoint $K_{1,1}$'s.*
2. *If $4 \mid m$, then $EUD(R)$ is the union of two disjoint $K_m$'s and $(m-2)/2$ disjoint $K_{1,1}$'s.*

*Proof.* Let $A = GF(p^n)$. For each $a \in U(A)$, let $X_a$ and $Y_a$ be as in the proof of Theorem 2.2. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude each $K_m$ of $UD(R)$ is a $K_1$ of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 2.2. $\square$

**Theorem 5.3.** *Let $n \geq 2$ be an integer, $R = Z_n \times Z_n$ and $\phi(n) = m$. Then*

1. *If $4 \mid n$, then $EUD(R)$ is the union of $m/2$ disjoint $K_{1,1}$'s.*

2. *If $4 \nmid n$ and $4 \nmid (p_i - 1)$ for at least one of the $p_i$'s in the prime factorization of $n$, then $EUD(R)$ is the union of $m/2$ disjoint $K_{1,1}$'s.*

3. *If $4 \nmid n$ and $4 \mid (p_i - 1)$ for all the odd $p_i$'s in the prime factorization of $n$, then we consider the two cases:*

   **Case I**. *If $n$ is even, then $EUD(R)$ is a union of $(m/2) - 2^{r-2}$ disjoint $K_{1,1}$'s and $2^{r-1}$ disjoint $K_1$'s.*

   **Case II**. *If $n$ is odd, then $EUD(R)$ is a union of $(m/2) - 2^{r-1}$ disjoint $K_{1,1}$'s and $2^r$ disjoint $K_1$'s.*

*Proof.* Let $A = Z_n$. For each $a \in U(A)$, let $X_a$ and $Y_a$ be as in the proof of Theorem 3.3. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude each $K_m$ of $UD(R)$ is a $K_1$ of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 3.3. $\square$

Let $R_1 = \{(u_1, z_1) \mid u_1 \in U(Z_n) \text{ and } z_1 \in Z(Z_n)\}$ and $R_2 = \{(z_2, u_2) \mid u_2 \in U(Z_n) \text{ and } z_2 \in Z(Z_n)\}$, see section 4. We define a relation $\sim$ on $R_1 \cup R_2$ such that $x \sim y$, where $x, y \in R_1 \cup R_2$, if $x = (c, c)y$ for some $(c, c) \in U(Z_n \times Z_n)$. It is clear that $\sim$ is an equivalence relation on $R_1 \cup R_2$. By construction of $R_1$ and $R_2$, it is clear that if $x \sim y$ for some $x, y \in R_1 \cup R_2$, then $x, y \in R_1$ or $x, y \in R_2$. Hence if $S$ is an equivalence class of $R_1 \cup R_2$, then there is an $a \in Z(Z_n)$ such that either $S = (\overline{(1, a)} = \{u(1, a) \mid u \in U(Z_n)\}$ or $S = \overline{(a, 1)} = \{u(a, 1) \mid u \in U(Z_n)\}$. Let $E(R_1 \cup R_2)$ be the set of all distinct equivalence classes of $R_1 \cup R_2$. We define the *equivalence zero-divisor dot product graph* $R_1 \cup R_2$ to be the (undirected) graph

$EZD(R_1 \cup R_2)$ with vertices $E(R_1 \cup R_2)$, and two distinct vertices $X$ and $Y$ are adjacent if and only if $a \cdot b = 0 \in A$ for every $a \in X$ and every $b \in Y$ (where $a \cdot b$ denote the normal dot product of $a$ and $b$). We have the following result.

**Theorem 5.4.** *Let $R = Z_n \times Z_n$ and $\phi(n) = m$. Then*

1. *If $n$ is prime, then $EZD(R_1 \cup R_2) = K_{1,1}$.*

2. *If $n$ is not prime, then $EZD(R_1 \cup R_2)$ is the union of of $(n - m)$ disjoint $K_{1,1}$'s.*

*Proof.* **(1)**. If $n$ is prime, then $E = \{\overline{(1,0)}, \overline{(0,1)}\}$. Thus $EZD(R_1 \cup R_2) = K_{1,1}$.

**(2)**. Suppose that $n$ is not prime, and let $A = Z_n$. For each $a \in Z(A)$, let $X_a$ and $Y_a$ be as in the proof of Theorem 4.1. Then $X_a, Y_a \in E(R_1 \cup R_2)$. Since $|X| = m$ for each $X \in E(R_1 \cup R_2)$, we conclude that each $K_{m,m}$ of $ZD(R_1 \cup R_2)$ is a $K_{1,1}$ of $EZD(R_1 \cup R_2)$. Hence the claim follows by the proof of Theorem 4.1. $\square$

**Remark 5.5.**

1. Let $A = Z_n$ and $R = Z_n \times Z_n$. Since for each $X \in E(U(R))$ there exists an $a \in U(A)$ such that $X = \overline{(1,a)} = \{u(1,a) \mid u \in U(A)\}$, note that we can recover the graph $UD(R)$ from the graph $EUD(R)$. However, drawing $EUD(R)$ is much simpler than drawing $UD(R)$.

2. Since for each $X \in E(R_1 \cup R_2)$ there exists an $a \in Z(Z_n)$ such that either $X = \overline{(1,a)} = \{u(1,a) \mid u \in U(Z_n)\}$ or $X = \overline{(a,1)} = \{u(a,1) \mid u \in U(Z_n)\}$, note that we can recover the graph $ZD(R_1 \cup R_2)$ from the graph $EZD(R_1 \cap R_2)$. However, drawing $EZD(R_1 \cup R_2)$ is much simpler than drawing $ZD(R_1 \cup R_2)$.

**Example 5.6.** Let $A = Z_{20}$ and $R = A \times A$. Then the $EUD(R)$ is the union of 4 disjoint $K_{1,1}$ by Theorem 5.3(1), and thus $UD(R)$ is the union of 4 disjoint $K_{8,8}$. The following is the graph of $EUD(R)$.

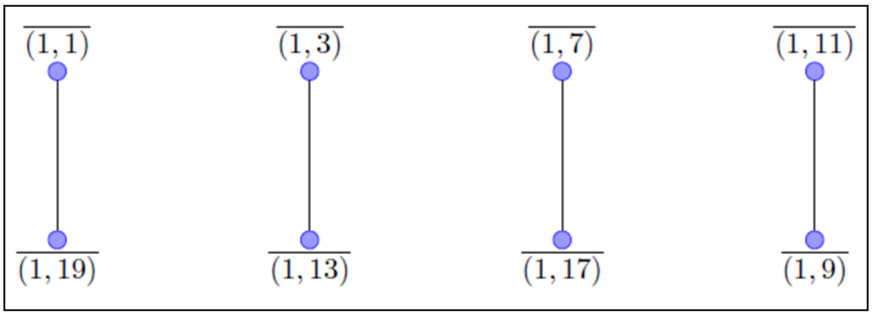


Fig. 5.1: The equivalence unit dot product graph of the ring $Z_{20} \times Z_{20}$

**Example 5.7.** Let $A = Z_{34}$ and $R = A \times A$. Then the $EUD(R)$ is the union of 7 disjoint $K_{1,1}$'s and 2 disjoint $K_1$'s by Theorem 5.3(3, Case I), and thus $UD(R)$ is the union of 7 disjoint $K_{16,16}$ and 2 disjoint $K_8$. The following is the graph of $EUD(R)$.

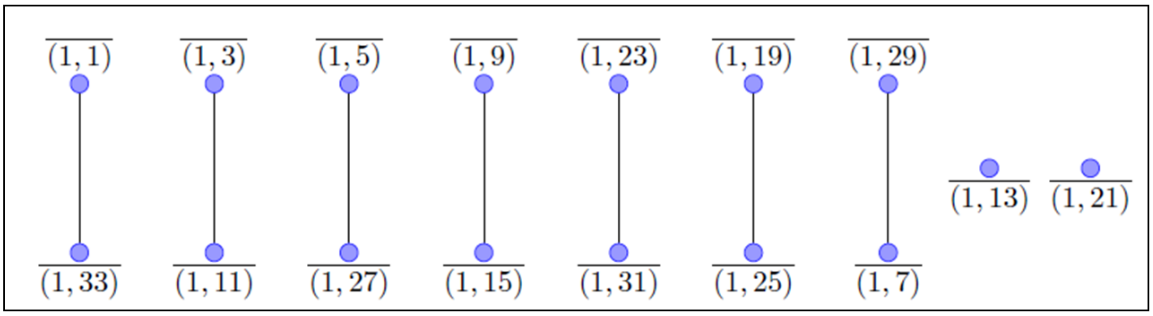


Fig. 5.2: The equivalence unit dot product graph of the ring $Z_{34} \times Z_{34}$

Correction:
Theorem 6.1 (1, 2, 3) and Theorem 6.2 (1, 2, 3): The set D is a dominating set and it is minimum in the sense that if F is another dominating set and F is a subset of D, then D = F. However, we cannot claim that D is a minimum dominating set. So the given dominating number \gamma need not be true.

Theorem 6.1 (4) and Theorem 6.2 (4): The result is OK here.

## 6. Domination Numbers of $TD(R)$, $ZD(R)$, and $UD(R)$

See the above note or my comments on the first page

Let $G$ be a graph with $V$ as its set of vertices. We recall that a subset $S \subseteq V$ is called a *dominating set of* $G$ if every vertex in $V$ is either in $S$ or is adjacent to a vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality among the dominating sets of $G$. Let $p$ be a positive prime number, $n \geq 1$. Then recall that $A = GF(p^n)$ denotes a finite field with $p^n$ elements.

**Theorem 6.1.** *Let $p$ be a positive prime integer, $n \geq 1$, $A = GF(p^n)$, and let $R = A \times \cdots \times A$ ($k$ times, where $k < \infty$). Then*

1. *$D = \{(1, a, 0, ..., 0) \mid a \in A\} \cup \{(0, 1, 0, ..., 0)\}$ is a minimal dominating set of $TD(R)$, and thus $\gamma(TD(R)) = p^n + 1$.*

2. *If $k = 2$, then $D = \{(1, 0), (0, 1)\}$ is a minimal dominating set of $ZD(R)$, and thus $\gamma(ZD(R)) = 2$.*

3. *If $k \geq 3$, then $D = \{(1, a, 0, ..., 0) \mid a \in A\} \cup \{(0, 1, 0, ..., 0)\}$ is a minimal dominating set of $ZD(R)$, and thus $\gamma(ZD(R)) = p^n + 1$.*

4. *If $k = 2$, then $D = \{(1, a) \mid a \neq 0\}$ is a minimal dominating set of $UD(R)$, and thus $\gamma(UD(R)) = p^n - 1$.*

*Proof.* **(1)**. Let $x = (x_1, x_2, ..., x_k)$ be a vertex in $TD(R)$. We consider two cases.

**Case I.** Assume that $x_2 \neq 0$. Then let $a = -x_1 x_2^{-1}$. Hence $v = (1, a, 0, ...., 0)$ is adjacent to $x$ in $TD(R)$.

**Case II**. Assume $x_2 = 0$. Then $v = (0, 1, 0...., 0) \in D$ is adjacent to $x$ in $TD(R)$. This shows that $D$ is a dominating set of $TD(R)$. In order to show it is minimal, let's first remove the vertex $w = (0, 1, 0...., 0)$. Then the vertex $v = (1, 0, 1...., 1)$ is not in the set $D$ and for every $d = (1, u, 0, ..., 0) \in D \setminus \{W\}$, we have $d \cdot v = (1, u, 0, ...., 0) \cdot (1, 0, 1...., 1) = 1 \neq 0$. Thus $w$ cannot be removed from $D$. Now assume that the vertex $m = (1, a, 0, ...., 0)$ is removed from D. Then $v = (a, -1, 0, ...., 0) \in TD(R)$, but $v$ is not adjacent to every $d \in D \setminus \{m\}$. Hence D is a minimal dominating set of $TD(R)$ and thus $\gamma(TD(R)) = |D| = p^n + 1$.

**(2)**. If $k = 2$ ,then the set of all non zero zero divisors of $R$ is $Z = \{(0, x) \mid x \in A^*\} \cup \{(y, 0) \mid y \in A^*\}$. Let v be a vertex in $ZD(R)$ if $v = (0, x)$ then it is connected to $(1, 0) \in D$ and if $v = (x, 0)$ then it is connected to $(0, 1) \in D$. This shows that D is a dominating set of $ZD(R)$. It is clear that D is in fact a minimal dominating set of $ZD(R)$ and hence $\gamma(ZD(R)) = |D| = 2$.

**(3)**. Assume that $k \geq 3$, and let $x = (x_1, x_2, ..., x_k)$ be a vertex in $ZD(R)$. Then at least one of the $x_i$'s is zero. (We will use similar argument as in (1)). We consider two cases.

**Case I.** Assume that $x_2 \neq 0$. Then let $a = -x_1 x_2^{-1}$. Hence $v = (1, a, 0, ...., 0)$ is adjacent to $x$ in $ZD(R)$.

**Case II**. Assume $x_2 = 0$. Then $v = (0, 1, 0...., 0) \in D$ is adjacent to $x$ in $ZD(R)$. This shows that $D$ is a dominating set of $ZD(R)$. In order to show it is minimal, let's first remove the vertex $w = (0, 1, 0...., 0)$. Then the vertex $v = (1, 0, 1...., 1)$ is not in the set $D$ and for every $d = (1, u, 0, ..., 0) \in D \setminus \{W\}$, we have $d \cdot v = (1, u, 0, ...., 0) \cdot (1, 0, 1...., 1) = 1 \neq 0$. Thus $w$ cannot be removed from $D$. Now assume that the vertex $m = (1, a, 0, ...., 0)$ is removed from D. Then $v = (a, -1, 0, ...., 0) \in TD(R)$, but $v$ is not adjacent to every $d \in D \setminus \{m\}$. Hence D is a minimal dominating set of $ZD(R)$ and thus $\gamma(ZD(R)) = |D| = p^n + 1$.

**(4)**. Let $x = (u_1, u_2)$ be a vertex in $UD(R)$ and assume that $x \notin D$. Let $a = -u_1 u_2^{-1}$. Then $(u_1, u_2)$ is adjacent to $(1, a) \in D$. Assume that $c = (1, a)$ is removed from $D$ for some $a \neq 0$. Then $(-a, 1)$ is not adjacent to every vertex in $D \setminus \{c\}$. Hence $D$ is a minimal dominating set and thus $\gamma(UD(R)) = |D| = p^n - 1$. $\square$

**Theorem 6.2.** *Let $n \geq 4$ be an integer that is not prime, $A = Z_n$, and $R = A \times \cdots \times A$ (k times, where $k < \infty$). Then write $n = p_1^{k_1} \cdots p_m^{k_m}$, where the $p_i$'s, $1 \leq i \leq m$, are distinct prime positive integers, and let $M = \{n/p_i \mid 1 \leq i \leq m\}$. Then*

1. *$D = \{(1, a, 0, ..., 0) \mid a \in A\} \cup \{(0, b, 0, ..., 0) \mid b \in M\}$ is a minimal dominating set of $TD(R)$, and thus $\gamma(TD(R)) = n + m$.*

2. *If $k = 2$, then $D = \{(0, a) \mid a \in M\} \cup \{(b, 0) \mid b \in M\}$ is a minimal dominating set of $ZD(R)$, and thus $\gamma(ZD(R)) = 2m$.*

3. *If $k \geq 3$, then $D = \{(1, a, 0, ..., 0) \mid a \in A\} \cup \{(0, b, 0, ..., 0) \mid b \in M\}$ is a minimal dominating set of $ZD(R)$, and thus $\gamma(ZD(R)) = n + m$.*

4. *If $k = 2$, then $D = \{(1, a) \mid a \in U(A)\}$ is a minimal dominating set of $UD(R)$, and thus $\gamma(UD(R)) = \phi(n)$.*

*Proof.*

**(1)**. Let $x = (x_1, x_2, ..., x_k)$ be a vertex in $TD(R)$. We consider two cases.

**Case I.** Assume that $x_2$ is a unit. Then let $a = -x_1 x_2^{-1}$. Hence $v = (1, a, 0, ...., 0)$ is adjacent to $x$ in $TD(R)$.

**Case II**. Assume $x_2$ is a zero-divisor of $A$. Then $p_i \mid x_2$ in $A$ for some $p_i$, $1 \leq i \leq m$. Then $v = (0, \frac{n}{p_i}, 0, ..., 0) \in D$ is adjacent to $x$ in $TD(R)$. This shows that $D$ is a dominating set of $TD(R)$. In order to show it is minimal, let's first remove the vertex $w = (0, \frac{n}{p_i}, 0, ..., 0)$ from $D$ for some $i$, $1 \leq i \leq m$. Then the vertex $v = (1, p_i, 1, ..., 1)$ is not in the set $D$ and for every $d = (1, u, 0, ..., 0) \in D \setminus \{W\}$, we have $d \cdot v = (1, u, 0, ...., 0) \cdot (1, p_i, 1...., 1) = 1 + up_i \neq 0$ (for if $1 + up_i = 0$, then $up_i = -1$ implies $p_i \in U(A)$, a contradiction). Thus $w$ cannot be removed from $D$. Now assume that the vertex $m = (1, a, 0, ...., 0)$ is removed from D. Then $v = (a, -1, 0, ...., 0) \in TD(R)$, but $v$ is not adjacent to every $d \in D \setminus \{m\}$. Hence D is a minimal dominating set of $TD(R)$ and thus $\gamma(TD(R)) = |D| = n + m$.

**(2)**. Let $x=(x_1,x_2)$ be a vertex in $ZD(R)$. Then $x_1 \in Z(A)$ or $x_2 \in Z(A)$. Assume that $x_1 \in Z(A)$. Hence $p_i \mid x_1$ in $A$ for some $i$, $1 \leq i \leq m$. Hence $x$ is adjacent to $(\frac{n}{p_i}, 0) \in D$. Assume that $x_2 \in Z(A)$. Hence $p_i \mid x_2$ in $A$ for some $i$, $1 \leq i \leq m$. Hence $x$ is adjacent to $(0, \frac{n}{p_i}) \in D$. In order to show that $D$ is minimal, let's first remove the vertex $w = (\frac{n}{p_i}, 0)$ from $D$ for some $i$, $1 \leq i \leq m$. Then $w=(p_i, 1)$ is a vertex of $ZD(R)$ and it is not adjacent to every vertex in $D \setminus \{w\}$. Assume that $m = (0, \frac{n}{p_i})$ is removed from $D$ for some $i$, $1 \leq i \leq m$. Then $(1, \frac{n}{p_i})$ is not adjacent to every vertex in $D \setminus \{m\}$. Thus $D$ is a minimal dominating set and hence $\gamma(ZD(R)) = 2m$.

**(3)**. Suppose that $k \geq 3$, and let $x = (x_1, x_2, ..., x_k)$ be a vertex in $ZD(R)$. Then at least one of the $x_i$'s is a zero-divisor of $A$. We consider two cases.

**Case I.** Assume that $x_2$ is a unit. Then let $a = -x_1x_2^{-1}$. Hence $v = (1, a, 0, ...., 0)$ is adjacent to $x$ in $ZD(R)$.

**Case II**. Assume $x_2$ is a zero-divisor of $A$. Then $p_i \mid x_2$ in $A$ for some $p_i$, $1 \leq i \leq m$. Then $v = (0, \frac{n}{p_i}, 0, ..., 0) \in D$ is adjacent to $x$ in $ZD(R)$. This shows that $D$ is a dominating set of $ZD(R)$. In order to show it is minimal, let's first remove the vertex $w = (0, \frac{n}{p_i}, 0, ..., 0)$ from $D$ for some $i$, $1 \leq i \leq m$. Then the vertex $v = (1, p_i, 1, ..., 1)$ is not in the set $D$ and for every $d = (1, u, 0, ..., 0) \in D \setminus \{W\}$, we have $d \cdot v = (1, u, 0, ...., 0) \cdot (1, p_i, 1...., 1) = 1 + up_i \neq 0$ (for if $1 + up_i = 0$, then $up_i = -1$ implies $p_i \in U(A)$, a contradiction). Thus $w$ cannot be removed from $D$. Now assume that the vertex $m = (1, a, 0, ...., 0)$ is removed from D. Then $v = (a, -1, 0, ...., 0) \in TD(R)$, but $v$ is not adjacent to every $d \in D \setminus \{m\}$. Hence D is a minimal dominating set of $ZD(R)$ and thus $\gamma(ZD(R)) = |D| = n + m$.

**(4)**. Let $x = (u_1, u_2)$ be a vertex in $UD(R)$. Let $x = (u_1, u_2)$ be a vertex in $UD(R)$ and assume that $x \notin D$. Let $a = -u_1u_2^{-1}$. Then $(u_1, u_2)$ is adjacent to $(1, a) \in D$. Assume that $c = (1, a)$ is removed from $D$ for some $a \in U(A)$. Then $(-a, 1)$ is not adjacent to every vertex in $D \setminus \{c\}$. Hence $D$ is a minimal dominating set and thus $\gamma(UD(R)) = |D| = \phi(n)$. $\square$

# 7. Conclusion and Future Work

In this thesis, we studied the unit dot product graph, the zero dot product graph and the total dot product graph of $Z_n \times Z_n$. We introduced a complete description of the structure of these graphs depending on the properties of the ring $Z_n$ .We started with the case where $n$ is a prime number,then we studied the more general case where $n$ is any positive integer.We proved that the structure of these graphs will vary depending on $n$. When we wanted to draw the unit dot product graph of $Z_n \times Z_n$ where $n$ is a large positive integer , It was useful to use the equivalence dot product graph whose set of vertices are equivalence classes.In chapter 5 ,we defined this new graph and introduced a description of its structure. In chapter 6, we determined the domination numbers of the unit dot product graph, zero dot product graph and total dot product graph for different values of $n$. In our future work,we are looking forward to generalizing our new theories to the case $R \times R$ where $R$ is any finite commutative ring using the new results and theories we proved in this thesis.

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# Vita

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